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Transformation between a hydrogen atom and a harmonic oscillator of arbitrary dimensions

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Abstract. Using the method of Zeng *et al.*, we discuss various problems with the relationship between a hydrogen atom and a harmonic oscillator of arbitrary dimensions, which include the laws of coordinate transformation, the transformations between the angular momenta, energies, radial functions, etc.

1. Introduction

Some research has been carried out on the relationship between a hydrogen atom and a harmonic oscillator of arbitrary dimensions. However, there are still many problems which need further study. For example, Davtyan and others [1–3] studied the coordinate transformation. Their method is to use a ‘definition’ to generalize to arbitrary dimension the relations satisfied for the so-called *KS* transformation [4–6]. They give no general proof for this extension. There are also some considerations in their works, which include the following meanings: the same function simultaneously represents the energy eigenstates of the two quantum systems; different energy levels of the hydrogen atom (or the harmonic oscillator) correspond to one energy level of the harmonic oscillator (or the hydrogen atom); etc. These considerations have no general significance, and raise some new questions. Bergmann and others [7, 8] studied the relationship between the radial functions. Their method is to make a formal comparison for the radial functions; they do not discuss whether the dimension, energy, angular momentum, etc, relations that they used are consistent with those derived from the coordinate transformation. Clearly, one needs a new method to study and solve these problems overall.

Recently [9], we used the coordinates of a hydrogen atom and the creation and annihilation operators of a harmonic oscillator to construct two sets of operators: they all form the $SU(1, 1)$ algebra, thus connecting the two quantum systems with each other. The two quantum systems are all of arbitrary dimension. Using this method, we established a simpler relationship between the two quantum systems, in which the energies and the energy eigenstates all have a one-to-one correspondence. Our method is based on the $SU(1, 1)$ algebra, and is therefore algebraic. It is very simple and effective. We believe that it may be used to solve all the problems of the transformation of the two quantum systems.

The relation between the energy eigenstates of the hydrogen atom and the harmonic oscillator given in [9] is in abstract form and is not expressed in coordinate space; moreover,

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it is only a part of all the problems which need solutions. In this paper, we will develop our method and apply it to give an overall discussion of the various problems on the transformation between the two quantum systems.

Let us now recall the method presented in [9]. We first construct the operators

$$\begin{aligned} K_1 &= \frac{1}{2}(x\Delta_x + x) \\ K_2 &= i\left(\frac{d-1}{2} + x_j \frac{\partial}{\partial x_j}\right) \\ K_3 &= -\frac{1}{2}(x\Delta_x - x) \end{aligned} \quad (1)$$

where x_i ($i = 1, 2, \dots, d$) are the coordinates of a d -dimensional hydrogen atom, $x = \sqrt{x_i x_i}$ and $\Delta_x = \partial^2/\partial x_i \partial x_i$. They were proved to form the $SU(1, 1)$ algebra, i.e. they satisfy

$$[K_1, K_2] = -iK_3 \quad [K_2, K_3] = iK_1 \quad [K_3, K_1] = iK_2. \quad (2)$$

We next construct the operators

$$\begin{aligned} K_1 &= \frac{1}{4} \sum_{\alpha=1}^D [(a_\alpha^\dagger)^2 + a_\alpha^2] \\ K_2 &= -\frac{i}{4} \sum_{\alpha=1}^D [(a_\alpha^\dagger)^2 - a_\alpha^2] \\ K_3 &= \frac{1}{4} \sum_{\alpha=1}^D (a_\alpha^\dagger a_\alpha + a_\alpha a_\alpha^\dagger) \end{aligned} \quad (3)$$

where a_α^\dagger and a_α ($\alpha = 1, 2, \dots, D$) are respectively the creation and annihilation operators of a D -dimensional harmonic oscillator. They were also proved to form the $SU(1, 1)$ algebra. Introducing the coordinates u_α of the D -dimensional harmonic oscillator and expressing a_α^\dagger and a_α in terms of u_α and $\partial/\partial u_\alpha$

$$a_\alpha = \frac{1}{\sqrt{2}} \left(u_\alpha + \frac{\partial}{\partial u_\alpha} \right) \quad a_\alpha^\dagger = \frac{1}{\sqrt{2}} \left(u_\alpha - \frac{\partial}{\partial u_\alpha} \right) \quad (4)$$

then equation (3) may be reduced to

$$\begin{aligned} K_1 &= \frac{1}{4}(\Delta_u + u^2) \\ K_2 &= \frac{i}{4} \left(D + 2u_\alpha \frac{\partial}{\partial u_\alpha} \right) \\ K_3 &= \frac{1}{4}(-\Delta_u + u^2) \end{aligned} \quad (5)$$

where $u = \sqrt{u_\alpha u_\alpha}$, $\Delta_u = \partial^2/\partial u_\alpha \partial u_\alpha$. The two sets of operators (1) and (5) form the $SU(1, 1)$ algebra; thus, the $SU(1, 1)$ algebra connects the two quantum systems, namely the d -dimensional hydrogen atom and the D -dimensional harmonic oscillator, with each other.

Noted that in this paper the variables x_i and u_α are assumed to be dimensionless.

The paper is organized as follows. In section 2, we derive the relation between the dimensions of the spaces of the two quantum systems and prove that the coordinate transformation between the two quantum systems forms a traceless Clifford algebra; no 'definition' or 'assumption' is used there. In section 3, we derive the relation between the angular momenta of the two quantum systems; There the concrete form of the coordinate transformation is not required, but rather only the relations that the coordinate transformation

should satisfy are used. In sections 4 and 5, we recall the relations between the energies and the energy eigenstates of the two quantum systems, and re-express them in coordinate space; then we use the angular momentum relation to derive the transformation between the radical functions. In all the results, the fact that the energies and the angular momenta of the two quantum systems have a one-to-one correspondence is demonstrated.

2. Transformation between coordinates

Since the two sets of operators (1) and (5) form the $SU(1, 1)$ algebra, identifying them with each other, we may establish the operator equations

$$x\Delta_x + x = \frac{1}{2}(\Delta_u + u^2) \tag{6}$$

$$\frac{d-1}{2} + x_i \frac{\partial}{\partial x_i} = \frac{1}{4} \left(D + 2u_\alpha \frac{\partial}{\partial u_\alpha} \right) \tag{7}$$

$$x\Delta_x - x = \frac{1}{2}(\Delta_u - u^2). \tag{8}$$

These equations are the theoretical fundamentals of this paper; of these, equations (6) and (8) may be associated with

$$x = \frac{1}{2}u^2 \tag{9}$$

$$x\Delta_x = \frac{1}{2}\Delta_u. \tag{10}$$

We now use the operator equations to derive the relations that the coordinate transformation between the d -dimensional hydrogen atom and the D -dimensional harmonic oscillator should satisfy. First, one notes that according to (9) x_i must be a homogeneous form of u_α with degree two: in other words, one should set

$$x_i = C_{\alpha\beta}^i u_\alpha u_\beta. \tag{11}$$

The coefficients $C_{\alpha\beta}^i$ in (11) may be considered to be the elements of some matrix C^i . They clearly satisfy $C_{\alpha\beta}^i = C_{\beta\alpha}^i$, i.e. the C^i are symmetric matrices.

Acting on (11) by the operator equation (7), one obtains:

Theorem 1. There is the following relation between the dimensions of the spaces of the d -dimensional hydrogen atom and the D -dimensional harmonic oscillator:

$$D = 2(d - 1). \tag{12}$$

The proof is straightforward, provided one notes that

$$x_i \frac{\partial x_j}{\partial x_i} = x_j \quad u_\alpha \frac{\partial}{\partial u_\alpha} (C_{\rho\sigma}^j u_\rho u_\sigma) = 2C_{\rho\sigma}^j u_\rho u_\sigma = 2x_j.$$

Theorem 1 shows that the dimension of the space of the D -dimensional harmonic oscillator connected by the d -dimensional hydrogen atom must be an even number.

Acting on (11) by the operator equation (10), one obtains:

Theorem 2. The matrices C^i , which associate the coordinates of the two quantum systems with each other, are those such that its traces are null:

$$C_{\alpha\alpha}^i = 0. \tag{13}$$

The proof is also straightforward, because equation (13) is shown easily from

$$0 = \Delta_u(C_{\alpha\beta}^i u_\alpha u_\beta).$$

We have shown that equation (9) means that x_i must takes the form (11). Further analysis for (9) and (11) results in:

Theorem 3. There are the following relations between the matrices C^i :

$$C^i C^j + C^j C^i = \frac{1}{2} I \delta_{ij} \quad (14)$$

where I is a unit matrix. Equation (14) means that the C^i form the Clifford algebra.

Proof. Differentiating $x = \frac{1}{2} u^2$ with respect to x_i and using (11) one may obtain

$$v_\alpha^i u_\alpha = 0 \quad (15)$$

where

$$v_\alpha^i = 2u^{-2} C_{\alpha\beta}^i u_\beta - \frac{\partial u_\alpha}{\partial x_i}. \quad (16)$$

There are two possibilities in (15): (i) $v_\alpha^i = 0$, (ii) $v_\alpha^i \neq 0$, $\mathbf{v}^i \perp \mathbf{u}$. We will prove $v_\alpha^i = 0$.

The \mathbf{v}^i are the vectors in the space of the D -dimensional harmonic oscillator, therefore, they may be expanded in terms of u_α . The expansion may be written as

$$v_\alpha^i = 2u^{-2} f_{\alpha\beta}^i u_\beta. \quad (17)$$

Substituting equation (17) in (15), one can easily see that the coefficients $f_{\alpha\beta}^i$ in (17) satisfy $f_{\alpha\beta}^i = -f_{\beta\alpha}^i$, i.e. the $f^i = (f_{\beta\alpha}^i)$ must be anti-symmetric matrices.

Differentiating (11) with respect to x_j , then expressing $\partial u_\alpha / \partial x_j$ in terms of v_α^j and expanding v_α^j in terms of u_α , one obtains

$$4u^{-2} (C^i C^j - C^i f^j)_{\alpha\beta} u_\alpha u_\beta = \delta_{ij}. \quad (18)$$

Setting $i = j$, equation (18) becomes

$$4u^{-2} (C^i C^i - C^i f^i)_{\alpha\beta} u_\alpha u_\beta = 1 \quad (\text{does not sum on } i). \quad (19)$$

From equation (19) one successively obtains

$$C^i C^i - C^i f^i = \frac{1}{4} I \quad (\text{does not sum on } i) \quad (20)$$

$$C^i - \frac{1}{4} (C^i)^{-1} = f^i. \quad (21)$$

The f^i are anti-symmetric, but the left-hand side of (21) is symmetric, therefore, one must have $f^i = 0$. This shows that $v_\alpha^i = 0$ and

$$\frac{\partial u_\alpha}{\partial x_i} = 2u^{-2} C_{\alpha\beta}^i u_\beta. \quad (22)$$

Since $f^i = 0$, one finds that the square of the matrix C^i is proportional to the unit matrix I :

$$C^i C^i = \frac{1}{4} I \quad (\text{does not sum on } i). \quad (23)$$

We now set $i \neq j$. In this case, equation (18) becomes

$$(C^i C^j)_{\alpha\beta} u_\alpha u_\beta = 0$$

which means that $(C^i C^j)_{\alpha\beta} = -(C^i C^j)_{\beta\alpha}$. Noting that $C_{\alpha\beta}^i = C_{\beta\alpha}^i$, one finds that the matrices C^i and C^j anti-commute with each other:

$$C^i C^j + C^j C^i = 0. \quad (24)$$

The association of (23) with (24) gives theorem 3, i.e. relation (14).

Theorems 1–3 give all the relations that the general coordinate transformation should satisfy. It is because the general coordinate transformation forms the Clifford algebra, therefore, that its concrete form may be obtained from the representation theory of the Clifford algebra [10, 11].

Note that equation (14) is not the same as that defined by other authors; therefore, the concrete form of the coordinate transformation in our work is not the same as that given by other authors (there is a difference of 2 in the constant time).

3. Transformation between angular momenta

We have used the operator equations (6)–(10) to determine the laws of the coordinate transformation. Conversely, when the coordinate transformation is given, one could also derive the operator equations (6)–(10).

As an example, we derive equation (10). From equations (9) and (22), one successively has

$$\begin{aligned} \frac{\partial}{\partial x_i} &= \frac{\partial u_\alpha}{\partial x_i} \frac{\partial}{\partial u_\alpha} = 2u^{-2} C_{\alpha\beta}^i u_\alpha \frac{\partial}{\partial u_\beta} \\ x \frac{\partial}{\partial x_i} &= C_{\alpha\beta}^i u_\alpha \frac{\partial}{\partial u_\beta} \\ x \frac{\partial}{\partial x_i} \left(x \frac{\partial}{\partial x_j} \right) &= C_{\alpha\beta}^i C_{\rho\sigma}^j u_\alpha \frac{\partial}{\partial u_\beta} \left(u_\rho \frac{\partial}{\partial u_\sigma} \right). \end{aligned}$$

The last of the above expressions may be reduced to

$$x_i \frac{\partial}{\partial x_i} + x^2 \Delta_x = \frac{d}{4} u_\alpha \frac{\partial}{\partial u_\alpha} + C_{\alpha\beta}^i C_{\rho\sigma}^j u_\beta u_\sigma \frac{\partial^2}{\partial u_\alpha \partial u_\rho}. \tag{25}$$

Differentiating x_i and x with respect to u_α , and using (9) and (11), one may show that

$$C_{\alpha\gamma}^i C_{\rho\sigma}^j u_\alpha u_\rho u_\sigma = \frac{1}{4} u^2 u_r. \tag{26}$$

The two sides of (26) include the variables u_α . Differentiating (26) three times, one obtains

$$C_{\omega r}^i C_{\sigma\rho}^j + C_{\sigma r}^i C_{\omega\rho}^j + C_{\rho r}^i C_{\omega\sigma}^j = \frac{1}{4} [\delta_{\sigma\omega} \delta_{\rho r} + \delta_{\sigma r} \delta_{\rho\omega} + \delta_{\rho\sigma} \delta_{r\omega}]. \tag{27}$$

Using equations (27) and (7), equation (25) may then be reduced to

$$x^2 \Delta_x = \frac{1}{4} u^2 \Delta_u + X \tag{28}$$

where the operator X is

$$X = -\frac{2-d}{4} u_\alpha \frac{\partial}{\partial u_\alpha} + \left\{ -C_{\rho\sigma}^i C_{\alpha\beta}^j - C_{\alpha\rho}^i C_{\beta\sigma}^j + \frac{1}{2} \delta_{\alpha\rho} \delta_{\beta\sigma} \right\} u_\rho u_\sigma \frac{\partial^2}{\partial u_\alpha \partial u_\beta}. \tag{29}$$

The operator X has a feature, i.e. its action on x_j gives the result 0:

$$X x_j = 0. \tag{30}$$

This result may be generalized to an arbitrary function $\Phi(\mathbf{x})$: $X\Phi(\mathbf{x}) = 0$, provided x_i in $\Phi(\mathbf{x})$ may be expressed in terms of u_α in the form (11), in other words, provided $\Phi(\mathbf{x})$ may be reduced to the function of u_α , i.e. one has $\Phi(\mathbf{x}) = \Psi'(u)$. Our operator equations always act on the function relation $\Phi(\mathbf{x}) = \Psi'(u)$. In studying the action of the operator equation (28), one always may set $X = 0$. In this case, equation (28) is reduced to

$$x \Delta_x = \frac{1}{2} \Delta_u. \tag{31}$$

From the coordinate transformation, one may indeed derive the operator equation (10).

We now study the relation between the angular momenta of the two quantum systems. Our conclusion is as follows:

Theorem 4. Let \hat{l}^2 and \hat{L}^2 respectively denote the squares of the angular momentum operators of the d -dimensional hydrogen atom and the D -dimensional harmonic oscillator, and l and L respectively denote the angular momenta (quantum numbers) of the d -dimensional hydrogen atom and the D -dimensional harmonic oscillator. There are the following relations between these operators and quantum numbers:

$$\hat{L}^2 = 4\hat{l}^2 \quad (32)$$

$$L = 2l. \quad (33)$$

Proof. We have used the coordinate transformation to derive (10). This operator equation provides us a simpler method to study the relation between \hat{l}^2 and \hat{L}^2 . We make the spherical surface S_x and S_u in the spaces of the two quantum systems respectively, and define the functions $Y(\mathbf{x})$ and $Y'(\mathbf{u})$ on S_x and S_u respectively. The radius of the spherical surfaces are x' and u' respectively and $x' = \frac{1}{2}u'^2$. $Y'(\mathbf{u})$ comes from the transformation of $Y(\mathbf{x})$, thus one has $Y(\mathbf{x}) = Y'(\mathbf{u})$.

In spherical coordinates, Δ_x and Δ_u have the forms

$$\begin{aligned} \Delta_x &= \frac{d^2}{dx^2} + \frac{d-1}{x} \frac{d}{dx} - \frac{\hat{l}^2}{x^2} \\ \Delta_u &= \frac{d^2}{du^2} + \frac{D-1}{u} \frac{d}{du} - \frac{\hat{L}^2}{u^2}. \end{aligned} \quad (34)$$

Acting on the function relation $Y(\mathbf{x}) = Y'(\mathbf{u})$ by the operator equation (10) and noting that $x' = \frac{1}{2}u'^2$, then one immediately finds that equation (32) holds.

According to angular momentum theory in high-dimensional space [12], the eigenvalues of \hat{l}^2 and \hat{L}^2 are $l(l+d-2)$ and $L(L+D-2)$, respectively. Noting $D = 2(d-1)$, one can see from (32) that (33) holds.

According to (32), \hat{l}^2 and \hat{L}^2 have common eigenfunctions. In other words, the eigenfunction of \hat{L}^2 (\hat{l}^2) may be obtained from the transformation of the eigenfunction of \hat{l}^2 (\hat{L}^2). We respectively use $Y_l(\theta_{d-2}, \theta_{d-1}, \dots, \theta_0)$ and $Y_L(\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0)$ to denote the eigenfunctions of \hat{l}^2 and \hat{L}^2 ; here $\theta_{d-2}, \theta_{d-1}, \dots, \theta_0$ and $\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0$ are respectively the angular coordinates of the d -dimensional hydrogen atom and the D -dimensional harmonic oscillator. Thus one has

$$Y_l(\theta_{d-2}, \theta_{d-1}, \dots, \theta_0) = M' Y_L(\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0) \quad (35)$$

where M' is a normalization constant, its introduction being due to the reason that $Y_l(\theta_{d-2}, \theta_{d-1}, \dots, \theta_0)$ and $Y_L(\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0)$ will be normalized in different spaces. $Y_l(\theta_{d-2}, \theta_{d-1}, \dots, \theta_0)$ and $Y_L(\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0)$ should be expressed in terms of the spherical harmonics in high-dimensional spaces, which may be found in [12]. In [13], we established a one-to-one correspondence between the spherical harmonics of a three-dimensional hydrogen atom and a four-dimensional harmonic oscillator, and solved all the problems of the transformation between these two quantum systems. One may generalize the method in [13] to study the relation between the spherical harmonics of a d -dimensional hydrogen atom and a D -dimensional harmonic oscillator, although the work is complicated. We do not discuss this problem in detail here.

4. Transformation between energy eigenfunctions

In [9] we have given the transformations between the energies and the energy eigenstates of the two quantum systems: they are

$$e_n = -\frac{2}{E_N^2} \tag{36}$$

$$|d, n\rangle = e^{-iK_2\theta_n} |D, N\rangle \tag{37}$$

where e_n and $|d, n\rangle$ are respectively the energy and the energy eigenstate of the d -dimensional hydrogen atom, E_N and $|D, N\rangle$ are respectively the energy and the energy eigenstate of the D -dimensional harmonic oscillator, and θ_n is the function defined by

$$\cosh \theta_n = \frac{1 - 2e_n}{\sqrt{-8e_n}} \quad \sinh \theta_n = -\frac{1 + 2e_n}{\sqrt{-8e_n}}. \tag{38}$$

The energies e_n ($n = 1, 2, 3, \dots$) and E_N ($N = 0, 1, 2, \dots$) are

$$e_n = -\frac{1}{2[n + \frac{1}{2}(d - 3)]^2} \quad \left(\text{in units of } \frac{\mu e^4}{\hbar^2}\right) \tag{39}$$

$$E_N = N + \frac{D}{2} \quad (\text{in units of } \hbar\omega). \tag{40}$$

Substituting equations (39) and (40) in (36) and noting $D = 2(d - 1)$, one obtains

$$N = 2(n - 1). \tag{41}$$

Equation (41) shows the relation between the energy levels of the two quantum systems. They are always in one-to-one correspondence. Moreover, the energy levels of the harmonic oscillator associated with the hydrogen atom are always even.

The energies of the two quantum systems are generally degenerate. For example, $|D, N\rangle$ and $|d, n\rangle$ should generally still include angular momentum quantum numbers. However, because the energy levels and the angular momenta of the two quantum systems are always in one-to-one correspondence, relation (37) always holds in any case.

We now transform (37) into coordinate space. Let $\Phi_{d,n,l}(\mathbf{x})$ denote the common eigenfunctions of the Hamiltonian and the operator \hat{l}^2 of the d -dimensional hydrogen atom, and $\Psi_{D,N,L}(\mathbf{u})$ the common eigenfunctions of the Hamiltonian and the operator \hat{L}^2 of the D -dimensional harmonic oscillator. It is clear that in coordinate space, equation (37) may be transformed into the form

$$\Phi_{d,n,l}(\mathbf{x}) = M e^{-iK_2\theta_n} \Psi_{D,N,L}(\mathbf{u}). \tag{42}$$

Because $\Phi_{d,n,l}(\mathbf{x})$ and $\Psi_{D,N,L}(\mathbf{u})$ will be normalized in different coordinate spaces, we have introduced a normalization constant M here. The operator K_2 in (42) must take the differential forms (1) or (5). A simple calculation shows that K_2 may also be reduced to

$$K_2 = i \left(\frac{d-1}{2} + x \frac{d}{dx} \right) = i \left(\frac{D}{4} + \frac{1}{2} u \frac{d}{du} \right). \tag{43}$$

The operator K_2 does not include angular coordinates, therefore, it commutes with \hat{l}^2 and \hat{L}^2 , i.e. it satisfies $[K_2, \hat{l}^2] = [K_2, \hat{L}^2] = 0$. Because of this fact, when $\Phi_{d,n,l}(\mathbf{x})$ is the eigenfunction of \hat{l}^2 (so it may be labelled by the angular momentum l), then $\Psi_{D,N,L}(\mathbf{u})$ must be the eigenfunction of \hat{L}^2 (so it may be labelled by the angular momentum L).

5. Transformation between radial functions

We now use spherical coordinates to write the eigenfunctions on both sides of (42), i.e. we set

$$\Phi_{d,n,l}(\mathbf{x}) = T_{d,n,l}(x)Y_l(\theta_{d-2}, \theta_{d-1}, \dots, \theta_0) \quad (44)$$

$$\Psi_{D,N,L}(\mathbf{u}) = R_{D,N,L}(u)Y_L(\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0) \quad (45)$$

where both $T_{d,n,l}(x)$ and $R_{D,N,L}(u)$ are radial functions. Noting that $\exp(-iK_2\theta_n)$ commutes with $Y_L(\varphi_{D-2}, \varphi_{D-1}, \dots, \varphi_0)$ and using (35), one can obtain

$$T_{d,n,l}(x) = M'' e^{-iK_2\theta_n} R_{D,N,L}(u) \quad (46)$$

which is simply the relation between the radial functions of the two quantum systems, where M'' is a normalization constant and satisfies $M = M'M''$.

The radial functions $T_{d,n,l}(x)$ and $R_{D,N,L}(u)$ are well known (one may find their forms from [8]):

$$T_{d,n,l}(y) = C(dnl)e^{-y/2}y^l L_{n-l-1}^{2l+d-2}(y) \quad (47)$$

$$C(dnl) = x_0^{-d/2} [n + \frac{1}{2}(d-3)]^{-(d+1)/2} [\Gamma(n-l)]^{\frac{1}{2}} [2\Gamma(n+l+d-2)]^{-\frac{1}{2}} \quad (48)$$

$$R_{D,N,L}(u) = C(DNL)e^{-u^2/2}u^L L_{(N-L)/2}^{(L+D/2-1)}(u^2) \quad (49)$$

$$C(DNL) = u_0^{-D/2} \left[2\Gamma\left(\frac{N-L}{2} + 1\right) \right]^{\frac{1}{2}} \left[\Gamma\left(\frac{N+L+D}{2}\right) \right]^{-\frac{1}{2}} \quad (50)$$

where $x_0 = \hbar^2/2me^2$, $u_0 = \sqrt{\hbar/m\omega}$, $y = 2x/n'$, $n' = n + \frac{1}{2}(d-3)$, and $L_m^{(\alpha)}(z)$ is the Laguerre polynomial.

Using the expressions for $T_{d,n,l}(x)$ and $R_{D,N,L}(u)$, one may check the correctness of relation (46) and determine the normalization constant M'' :

$$M'' = x_0^{-d/2} u_0^{D/2} [n + \frac{1}{2}(d-3)]^{-1}. \quad (51)$$

In checking relation (46), one needs to expand the exponent functions on the right-hand side of (46) as series and use

$$e^{-iK_2\theta_n} = e^{\frac{1}{4}D\theta_n} e^{\frac{1}{2}\theta_n u d/du} \quad (52)$$

$$L_n^{(\alpha)}(z) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{z^k}{k!} \quad (53)$$

$$\left(u \frac{d}{du}\right)^l u^a = a^l u^a \quad (54)$$

$$e^{\theta_n} = \frac{1}{n'} \quad n' = n + \frac{1}{2}(d-3). \quad (55)$$

In this way, the right-hand side of (46) would be changed to the same form as its left-hand side. All the calculations are in fact simpler.

6. Conclusion

We have discussed all the problems on the transformation between a hydrogen atom and a harmonic oscillator of arbitrary dimensions. Our notion and method are not the same as that used by other authors, but are very strict. The results are all correct and clear up many problems that appeared in the previous work cited herein.

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